Economic Growth Toolbox

1° Dynamic optimization (with continuous time & infinite time horizon)

2° Monopolistic competition (à la Dixit-Stiglitz)
(R&D based models feature increasing returns to scale, which are inconsistent with perfect competition)

3° General equilibrium

4° Comparisons: Decentralized Equilibrium vs. Social Planner

The most basic dynamic optimization problem in Growth Theory

- 'dynastic model', $t \in [0, +\infty)$
- infinite horizon, discounting
- the consumption–savings decision of the household

c – control variable /CONSUMPTION/

a – state variable /ASSETS; CAPITAL/, $\dot{a} = \frac{da}{dt}$

$$\max_{\{c(t)\}_{t=0}^{\infty}} \int_0^{+\infty} e^{-\rho t} u(c(t))\, dt \quad \text{s.t.} \quad \dot{a} = ra + w - c$$

we'd like to find

- the Euler equation $\dot{c} = \ldots$
- ideally the optimal path $c(t) = \ldots$, $a(t) = \ldots$

Solution:

1° Define Hamiltonian

$$\mathcal{H}(c, a; \lambda) = \underbrace{e^{-\rho t} u(c)}_{\text{TERM WITHIN INTEGRAL}} + \lambda \underbrace{(ra + w - c)}_{\text{RHS OF EQ. OF MOTION } \dot{a} = \dots}$$

↑ CO-STATE VARIABLE
SHADOW PRICE OF a

2° Pontryagin maximum principle

- $\frac{\partial \mathcal{H}}{\partial c} = 0 \qquad \left(\max_c \mathcal{H}\right)$
- $\frac{\partial \mathcal{H}}{\partial a} = -\dot{\lambda}$
- $\frac{\partial \mathcal{H}}{\partial \lambda} = \dot{a}$

Here,
$$\begin{cases} e^{-\rho t} u'(c) - \lambda = 0 \\ \lambda r = -\dot{\lambda} \\ \dot{a} = ra + w - c \end{cases}$$

Trick: use log derivatives,
$\hat{x} = \frac{\dot{x}}{x} = \frac{\partial(\ln x)}{\partial t}$ if $x > 0$.

Rule:
when $x = a^{\alpha} b^{\beta}$,
$\hat{x} = \alpha \hat{a} + \beta \hat{b}$

3° Solve for Euler equation

$$\begin{cases} \lambda = e^{-\rho t} u'(c) \Rightarrow \hat{\lambda} = -\rho + \widehat{u'(c)} \Rightarrow \hat{\lambda} = -\rho + \frac{u''(c)\dot{c}}{u'(c)} \\ \hat{\lambda} = -r \\ \dot{a} = ra + w - c \end{cases}$$

$$\Rightarrow -r = -\rho + \underbrace{\frac{u''(c)c}{u'(c)}}_{-\theta(c)} \cdot \underbrace{\frac{\dot{c}}{c}}_{\hat{c}} \Rightarrow \boxed{\hat{c} = \frac{r - \rho}{\theta(c)}}$$

! Euler equation

4° Use TVC: transversality conditions (also part of the Pontryagin maximum principle)

$$\begin{cases} \lim_{t\to\infty} \lambda(t) = 0 \\ \lim_{t\to\infty} \mathcal{H}(t) = 0 \end{cases}$$

[We also frequently use (instead) a stronger, single TVC:

$$\lim_{t\to\infty} \lambda(t)a(t) = 0$$]

[When $\underbrace{\lim_{t\to\infty} \widehat{\lambda(t)a(t)} < 0}_{\text{OFTEN SUFFICES IN GROWTH THEORY.}}$ then $\lim_{t\to\infty} \lambda(t)a(t) = 0$.]

Here: It depends on the assumptions on $r(t)$, $w(t)$.

EQUIVALENT APPROACHES

Note:

Hamiltonian	Current-value Hamiltonian
$\mathcal{H} = e^{-\rho t}u(c) + \lambda(ra + w - c)$	$\mathcal{H} = u(c) + \mu(ra + w - c)$
FOCs:	FOCs:
$\frac{\partial \mathcal{H}}{\partial c} = 0$	$\frac{\partial \mathcal{H}}{\partial c} = 0$
$\frac{\partial \mathcal{H}}{\partial a} = -\dot{\lambda}$	$\frac{\partial \mathcal{H}}{\partial a} = \rho\mu - \dot{\mu}$
$\frac{\partial \mathcal{H}}{\partial \lambda} = \dot{a}$	$\frac{\partial \mathcal{H}}{\partial \lambda} = \dot{a}$